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Igusa Local Zeta Functions and Parabolic Castling Transformation of Prehomogeneous Vector Spaces

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between the Igusa local zeta function of a prehomogeneous vector space and the Igusa local zeta function of its castling transform. Y. Teranishi gives one generalization of the castling transformation. This generalized castling transformation is related to parabolic subgroups, hence we call it the parabolic castling transformation. In this paper, we give a relation between the Igusa local zeta function of a prehomogeneous vector space and the Igusa local zeta function of its parabolic castling transform.

Contents.

INTRODUCTION

The castling transformation is a standard procedure for constructing new prehomogeneous vector spaces from a given one and plays a crucial role in the classification of prehomogeneous vector spaces (cf. [6, Sect. 2]). Hence it is quite important to relate properties of a prehomogeneous vector space to the corresponding properties of its castling transform. In particular, Professor J.-I. Igusa gives a result of this kind; he finds a relation between the Igusa local zeta function of a prehomogeneous vector space and the Igusa local zeta function of its castling transform (cf. [2, Sect. 8])—the

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Igusa local zeta function is one of the local zeta functions over a $\wp$-adic field K; the following is one of the simplest examples,

$$Z_K(s)=\int_{x\in \mathcal{O}_K}|x|_K^s\,dx=\frac{1-q^{-1}}{1-q^{-(s+1)}}\qquad (s\in \mathbf{C},\ \mathrm{Re}(s)>0),$$

for which the definitions will be given in Section 1.

In [7], Y. Teranishi gives one generalization of the castling transformation. This generalized castling transformation is related to parabolic subgroups; hence we call it he *parabolic castling transformation*. It is a natural problem to extend Igusa's result to a relation between the Igusa local zeta function of a prehomogeneous vector space (G, ρ, V) and the Igusa local zeta function of its parabolic castling transform $(\tilde{G}, \tilde{\rho}, \tilde{V})$. This is exactly the problem we consider in this paper. Here we give a brief sketch of our result, which will be stated in Section 2, precisely. Let $Z_K(\omega_s\chi)=Z_K(\omega_{s_1}\chi_1, ..., \omega_{s_\ell}\chi_\ell)$ and $\tilde{Z}_K(\omega_s\chi)=\tilde{Z}_K(\omega_{s_1}\chi_1, ..., \omega_{s_\ell}\chi_\ell)$, respectively, be the Igusa local zeta functions of (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$. Then we have the following formula, which is the main result of this paper,

$$\frac{Z_K(\omega_s\chi)}{\prod_k((1-q^{-b_k})/(1-q^{-(a_k(s)+b_k)}))}=\frac{\tilde{Z}_K(\omega_s\chi)}{\prod_k((1-q^{-\tilde{b}_k})/(1-q^{-(\tilde{a}_k(s)+\tilde{b}_k)}))},$$

in which $a_k(s)$ and $\tilde{a}_k(s)$ are some linear combinations of $s_1, ..., s_\ell$, whose coefficients are positive integers, and b_k and $\tilde{b}_k$ are also some positive integers. Here we should emphasize the following points

(1) $Z_K(\omega_s\chi)$ and $\tilde{Z}_K(\omega_s\chi)$ coincide with each other, up to a product of some standard factors—standard factors are rational functions of $q^{-s_1}, ..., q^{-s_\ell}$ which are similar to the above simplest Igusa local zeta function.

(2) The products of the standard factors which appear in our formula correspond to the Young diagrams determined by the parabolic subgroups and the degrees of the basic relative invariants of (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

1. PRELIMINARIES

1.1. *Igusa Local Zeta Functions*

Let K be a $\wp$-adic number field, namely a finite extension of the p-adic number field $\mathbf{Q}_p$. We denote by $\bar{K}$ the algebraic closure of K. Let $\mathcal{O}_K$ be the ring of integers in K. We fix a prime element π_K once and for all, then $\mathcal{P}_K=\pi_K\mathcal{O}_K$ is the unique maximal ideal of $\mathcal{O}_K$. We denote by $\mathcal{O}_K^\times$ the unit group of $\mathcal{O}_K$, namely $\mathcal{O}_K^\times = \mathcal{O}_K - \mathcal{P}_K$. Every $\alpha \in K^\times$ can be uniquely expressed as

$$\alpha = \pi_K^{\mathrm{ord}_K(\alpha)}\, \mathrm{ac}(\alpha)$$

with $\mathrm{ord}_K(\alpha) \in \mathbf{Z}$ and $\mathrm{ac}(\alpha) \in \mathcal{O}_K^\times$. The cardinality of the residue field $\mathcal{O}_K/\mathcal{P}_K$ is denoted by q,

$$q = \#(\mathcal{O}_K/\mathcal{P}_K),$$

and we normalize the absolute value of $\alpha \in K^\times$ by

$$|\alpha|_K = q^{-\mathrm{ord}_K(\alpha)}.$$

Let $\Omega(K^\times)$ be the group of quasi-characters of $K^\times$. For a complex number $s \in \mathbf{C}$, we define $\omega_s \in \Omega(K^\times)$ by

$$\omega_s(\alpha) = |\alpha|_K^s \qquad (\alpha \in K^\times).$$

We denote by $\widehat{\mathcal{O}_K^\times}$ the dual group of $\mathcal{O}_K^\times$. We identify a $\chi \in \widehat{\mathcal{O}_K^\times}$ with the character of $K^\times$ obtained by expanding χ to $K^\times$ so that $\chi(\pi_K) = 1$. Every $\omega \in \Omega(K^\times)$ can be expressed as

$$\omega = \omega_s \chi$$

with

$$s \in \mathbf{C} \Big/ \left(\frac{2\pi\sqrt{-1}}{\log q}\right) \mathbf{Z} \qquad \text{and} \qquad \chi \in \widehat{\mathcal{O}_K^\times},$$

and we put

$$\mathrm{Re}(\omega) = \mathrm{Re}(s) \qquad (s = \log \omega(\pi_K)/\log q).$$

For an $s = (s_1, ..., s_\ell) \in \mathbf{C}^\ell$ and a $\chi = (\chi_1, ..., \chi_\ell) \in (\widehat{\mathcal{O}_K^\times})^\ell$, we put

$$\omega_s \chi = (\omega_{s_1}\chi_1, ..., \omega_{s_\ell}\chi_\ell) \in \Omega(K^\times)^\ell.$$

Let V_K be a finite dimension K-vector space and $K[V_K]$ the ring of K-polynomials on V_K. For an $\omega = (\omega_1, ..., \omega_\ell) \in \Omega(K^\times)^\ell$ and an $f(x) = (f_1(x), ..., f_\ell(x)) \in K[V_K]^\ell$, we put

$$\omega(f(x)) = \omega_1(f_1(x)) \cdots \omega_\ell(f_\ell(x)).$$

We denote by dx the Haar measure o V_K normalized by $\mathrm{vol}(V_K(\mathcal{O}_K)) = \int_{V_K(\mathcal{O}_K)} dx = 1$. Take an $f(x) = (f_1(x), ..., f_\ell(x)) \in K[V_K]^\ell$, we define the *Igusa local zeta function* $Z_K(\omega)$ of $f(x)$ as

$$Z_K(\omega) = \int_{V_K(\mathcal{O}_K)} \omega(f(x))\, dx.$$

It is clear that $Z_K(\omega)$ is absolutely convergent for $\mathrm{Re}(\omega_J) > 0 (1 \leqslant j \leqslant \ell)$ and represents a holomorphic function on $\{\omega \in \Omega(K^\times)^\ell \mid \mathrm{Re}(\omega_j) > 0\ (1 \leqslant j \leqslant \ell)\}$. Moreover, the following lemma is well-known.

Lemma 1.1 [5, Lemma 2.1; 1, Theorem A]. *The Igusa local zeta function $Z_K(\omega)$ has an analytic continuation to a meromorphic function on $\Omega(K^\times)^\ell$. Moreover, for each $\chi = (\chi_1, ..., \chi_\ell) \in (\widehat{\mathcal{O}_K^\times})^\ell$, there exists a collection of integers*

$$\{a_1^{(k)}, ..., a_\ell^{(k)}, b^{(k)}, m_k \mid 1 \leqslant k \leqslant h, 0, \leqslant m_k \leqslant [K : \mathbf{Q}_p] \cdot \dim V_K\}$$

and a polynomial $P_\chi(q^{\pm s_1}, ..., q^{\pm s_\ell})$ in $q^{\pm s_1}, ..., q^{\pm s_\ell}$ such that

$$Z_K(\omega_s \chi) = P_\chi(q^{\pm s_1}, ..., q^{\pm s_\ell}) / \prod_{k=1}^{h} (1 - q^{-\sum_{i=1}^{\ell} a_i^{(k)} s_i - b^{(k)}})^{m_k}.$$

1.2. *Prehomogeneous Vector Spaces and Their Relative Invariants*

Let G be a connected linear algebraic group defined over K, V a finite dimensional $\bar{K}$-vector space with K-structure V_K, and $\rho: G \to GL(V)$ a K-rational representation of G on V. Then the triplet (G, ρ, V) is a *prehomogeneous vector space defined over K* if there exists a proper algebraic subset S of V such that $V - S$ is a single $\rho(G)$-orbit. The algebraic set S is called the *singular set* of (G, ρ, V) and is also defined over K (cf. [4, Lemma 1.1]).

A non-zero K-rational function $f(x)$ on V is called a *K-relative invariant* of (G, ρ, V) if there exists a K-rational character v of G such that

$$f(\rho(g)\, x) = v(g)\, f(x) \qquad (g \in G, x \in V).$$

Let S_j $(1 \leqslant j \leqslant \ell)$ be the K-irreducible hypersurfaces contained in the singular set S. For each j $(1 \leqslant j \leqslant \ell)$, we take the K-irreducible polynomial $f_j(x)$ defining S_j; then it is well known that $f_1(x), ..., f_\ell(x)$ are K-relative invariants of (G, ρ, V) and any K-relative invariant can be uniquely expressed as

$$c \cdot f_1(x)^{\mu_i} \cdots f_\ell(x)^{\mu_\ell} \qquad (c \in K^\times, \mu_1, ..., \mu_\ell \in \mathbf{Z})$$

(cf. [4, Lemma 1.3]). The polynomials $f_1(x), ..., f_\ell(x)$ are called the *K-basic relative invariants* of (G, ρ, V) and ℓ is called the *K-rank* of (G, ρ, V).

In this paper, we consider the Igusa local zeta function of the K-basic relative invariants of (G, ρ, V), and call it the *Igusa local zeta function* of (G, ρ, V).

2. MAIN THEOREM

2.1. *Parabolic Castling Transformation*

At first, we recall the definition of the parabolic castling transformation. Let H be a connected linear algebraic group defined over K and $\rho_1 = H \to GL_n(\bar{K})$ and n-dimensional K-rational representation of H on $GL_n(\bar{K})$. Throughout this paper, we fix an ordered partition $e_1 + \cdots + e_r = n$ $(r \geqslant 2)$. Let $P_{e_1, ..., e_{r-1}}$ be the standard parabolic subgroup of the general linear group $GL_{n-e_r}(\bar{K})$ associated to the ordered partition $e_1 + \cdots + e_{r-1} = n - e_r$, namely the subgroup of $GL_{n-e_r}(\bar{K})$ consisting of all matrices of the form

$$\begin{pmatrix} p_1 & p_{12} & \cdots & p_{1\,r-1} \\ & p_2 & & \vdots \\ & & \ddots & \vdots \\ 0 & & & p_{r-1} \end{pmatrix},$$

where $p_i \in GL_{e_i}(\bar{K})$ $(1 \leqslant i \leqslant r-1)$ and $p_{ij} \in M_{e_i, e_j}(\bar{K})$ $(1 \leqslant i < j \leqslant r-1)$. Similarly, let $P_{e_r, ..., e_2}$ be the standard parabolic subgroup of the general linear group $GL_{n-e_1}(\bar{K})$ associated to the ordered partition $e_r + \cdots + e_2 = n - e_1$. Put $G = H \times P_{e_1, ..., e_{r-1}}$ and $V = M_{n, n-e_r}(\bar{K})$. Also put $\tilde{G} = H \times P_{e_r, ..., e_2}$ and $\tilde{V} = M_{n, n-e_1}(\bar{K})$. Moreover, put $V_K = M_{n, n-e_r}(K)$ and $\tilde{V} = M_{n, n-e_1}(K)$; then V_K and $\tilde{V}_K$ are K-structures of V and $\tilde{V}$, respectively. Let ρ: $G \to GL(V)$ be the K-rational representation of G on V defined by

$$\rho(h, p)\, x = \rho_1(h)\, x p^{-1} \qquad ((h, p) \in G, x \in V).$$

Similarly, let $\tilde{\rho} = \tilde{G} \to GL(\tilde{V})$ denote the K-rational representation of $\tilde{G}$ on $\tilde{V}$ defined by

$$\tilde{\rho}(h, \tilde{p})\, \tilde{x} = {}^t\rho_1(h)^{-1}\, \tilde{x}\tilde{p}^{-1} \qquad ((h, \tilde{p}) \in \tilde{G}, \tilde{x} \in \tilde{V}).$$

The triplet $(\tilde{G}, \tilde{\rho}, \tilde{V})$ is called the parabolic castling transform of (G, ρ, V) and vice versa. The following lemma is wellknown.

LEMMA 2.1 [7, Lemma 1.3; 3, Proposition 1.3 and Corollary 1.4]. *The triplet* (G, ρ, V) *is a prehomogeneous vector space if and only if so is its parabolic castling transform* $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

Remark. If $r = 2$, then Lemma 2.1 gives the usual castling transformation.

Throughout of this paper, we assume that

(G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$ are prehomogeneous vector spaces defined over K.

Now, we show that there exists a one-to-one corresponding between the K-basic relative invariants of (G, ρ, V) and the K-basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

Let $\lambda = (\lambda_1, ..., \lambda_m)$ be a sequence of positive integers satisfying $1 \leqslant \lambda_1 < \cdots < \lambda_m \leqslant n$, where n, m are positive integers such that $n \geqslant m$. For a matrix $A \in M_{n, m}(\bar{K})$, we denote by $\det_\lambda(A)$ the determinant of the square matrix of degree m with the λ_k th row of A as its kth row $(1 \leqslant k \leqslant m)$.

For the ordered partition $e_1 + \cdots + e_r = n$, we put $d_i = e_1 + \cdots + e_i$, $d'_i = n - d_i = e_{i+1} + \cdots + e_r$ $(1 \leqslant i \leqslant r-1)$. For each i $(1 \leqslant i \leqslant r-1)$, we put

$$\Lambda_i = \{\lambda = (\lambda_1, ..., \lambda_{d_i}) \in \mathbf{N}^{d_i} \mid 1 \leqslant \lambda_1 < \cdots < \lambda_{d_i} \leqslant n\}$$

and

$$\tilde{\Lambda}_i = \{\tilde{\lambda} = (\tilde{\lambda}_{d_i+1}, ..., \tilde{\lambda}_n) \in \mathbf{N}^{d'_i} \mid 1 \leqslant \tilde{\lambda}_{d_i+1} < \cdots < \tilde{\lambda}_n \leqslant n\}.$$

The cardinalities of Λ_i and $\tilde{\Lambda}_i$ $(1 \leqslant i \leqslant r-1)$ coincide with each other and we put $m_i = \#\Lambda_i = \#\tilde{\Lambda}_i = \binom{n}{d_i}$.

For each $\lambda = (\lambda_1, ..., \lambda_{d_i}) \in \Lambda_i$ $(1 \leqslant i \leqslant r-1)$, there uniquely exists a $\lambda^* = (\tilde{\lambda}_{d_i+1}, ..., \tilde{\lambda}_n) \in \tilde{\Lambda}_i$ such that $\{\lambda_1, ..., \lambda_{d_i}, \tilde{\lambda}_{d_i+1}, ..., \tilde{\lambda}_n\} = \{1, ..., n\}$ as sets. The mapping $\lambda \mapsto \lambda^*$ gives a one-to-one correspondence from Λ_i to $\tilde{\Lambda}_i$. We denote by $\operatorname{sgn}(\lambda)$ the signature of the permutation

$$\begin{pmatrix} 1 & \cdots & d_i & d_{i+1} & \cdots & n \\ \lambda_1 & \cdots & \lambda_{d_i} & \tilde{\lambda}_{d_i+1} & \cdots & \tilde{\lambda}_n \end{pmatrix}.$$

For a matrix $x \in V$, we denote by x^{d-i} $(1 \leqslant i \leqslant r-1)$ the $n \times d_i$ matrix defined by $x = (x^{d_i} \mid *)$. Similarly, for a matrix $\tilde{x} \in \tilde{V}$, we denote by $\tilde{x}^{d'_i}$ $(1 \leqslant i \leqslant r-1)$ the $n \times d'_i$ matrix defined by $\tilde{x} = (\tilde{x}^{d'_i} \mid *)$.

Take matrices $x \in V$ and $\tilde{x} \in \tilde{V}$. Then we have, for each i $(1 \leqslant i \leqslant r-1)$,

$$\det(x^{d_i} \mid \tilde{x}^{d'_i}) = \sum_{\lambda \in \Lambda_i} \det_\lambda(x^{d_i}) \cdot \operatorname{sgn}(\lambda) \det_{\lambda^*}(\tilde{x}^{d'_i}).$$

Let $\Delta_i\colon V \to \bar{K}^{m_i}$ and $\tilde{\Delta}_i\colon \tilde{V} \to \bar{K}^{m_i}$, respectively, be the mappings defined by

$$\Delta_i(x) = (\det_\lambda (x^{d_i}))_{\lambda \in \Lambda_i} \qquad (x \in V)$$

and

$$\tilde{\Delta}_i(\tilde{x}) = (\operatorname{sgn}(\lambda) \det_{\lambda^*} (\tilde{x}^{d'_i}))_{\lambda \in \Lambda_i} \qquad (\tilde{x} \in \tilde{V}).$$

Moreover, we put

$$\Delta\colon V \to \bar{K}^{m_1} \oplus \cdots \oplus \bar{K}^{m_{r-1}}, \qquad \Delta(x) = (\Delta_i(x))_{1 \leqslant i \leqslant r-1} \qquad (x \in V)$$

and

$$\tilde{\Delta}\colon \tilde{V} \to \bar{K}^{m_1} \oplus \cdots \oplus \bar{K}^{m_{r-1}}, \qquad \tilde{\Delta}(\tilde{x}) = (\tilde{\Delta}_i(\tilde{x}))_{1 \leqslant i \leqslant r-1} \qquad (\tilde{x} \in \tilde{V}).$$

Let $f(x)$ be a K-relative invariant polynomial of (G, ρ, V). Then, by the first main theorem for the parabolic subgroup $P_{e_1, \ldots, e_{r-1}}$, there exists a K-polynomial $F(z)$ on $\bar{K}^{m_1} \oplus \cdots \oplus \bar{K}^{m_{r-1}}$ satisfying $f(x) = F(\Delta(x))$. We define a K-polynomial $\tilde{f}(\tilde{x})$ on $\tilde{V}$ by $\tilde{f}(\tilde{x}) = F(\tilde{\Delta}(\tilde{x}))$, then $\tilde{f}(\tilde{x})$ is a K-relative invariant polynomial of $(\tilde{G}, \tilde{\rho}, \tilde{V})$. The mapping $f(x) \mapsto \tilde{f}(\tilde{x})$ induces a one-to-one correspondence between the K-relative invariants of (G, ρ, V) and the K-basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$. Note that $f(x)$ is irreducible if and only if $\tilde{f}(\tilde{x})$ is irreducible. Therefore we have the following lemma.

LEMMA 2.2. (1) *The K-rank of (G, ρ, V) is equal to the K-rank of $(\tilde{G}, \tilde{\rho}, \tilde{V})$. We denote by ℓ the K-rank of (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$.*

(2) *There exist irreducible K-polynomials $F_1(z), \ldots, F_\ell(z)$ on $\bar{K}^{m_1} \oplus \cdots \oplus \bar{K}^{m_{r-1}}$ such that*

$$F_1(\Delta(x)), \ldots, F_\ell(\Delta(x))$$

are the K-basic relative invariants of (G, ρ, V) and

$$F_1(\tilde{\Delta}(\tilde{x})), \ldots, F_\ell(\tilde{\Delta}(\tilde{x}))$$

are the K-basic relative invariants of $(\tilde{G}, \tilde{\rho}, \tilde{V})$.

Remark. If we write $F_j(z) = F_j(z_1^1, \ldots, z_{m_1}^1, \ldots, z_1^{r-1}, \ldots, z_{m_{r-1}}^{r-1})$ $(1 \leqslant j \leqslant \ell)$, then, for each i $(1 \leqslant i \leqslant r-1)$, $F_j(z)$ is homogeneous with respect to m_i-variables $z^i = (z_1^i, \ldots, z_{m_i}^i)$; we denote by $\deg_i(F_j)$ the homogeneous degree of $F_j(z)$ with respect to z^i.

2.2. *Main Theorem*

Now we state the main theorem.

Let $Z_K(\omega)$ and $\tilde{Z}_K(\omega)$ be the Igusa local zeta functions of (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$, respectively, namely the meromorphic functions of $\omega = (\omega_1, ..., \omega_\ell) \in \Omega(K^\times)^\ell$ defined by

$$Z_K(\omega) = \int_{x \in M_{n,\, n-e_r}(\mathcal{O}_K)} \omega(F(\Delta(x)))\, dx$$

and

$$\tilde{Z}_K(\omega) = \int_{\tilde{x} \in M_{n,\, n-e_1}(\mathcal{O}_K)} \omega(F(\tilde{\Delta}(\tilde{x})))\, d\tilde{x},$$

for $\mathrm{Re}(\omega_j) > 0$ $(1 \leqslant j \leqslant \ell)$, in which we put $F(z) = (F_1(z), ..., F_\ell(z))$.

We define the positive integers ξ_k^j $(1 \leqslant j \leqslant \ell,\ 1 \leqslant k \leqslant n - e_r)$ as follows:

$$\xi_1^j = \cdots = \xi_{d_1}^j = \deg_1(F_j) + \cdots + \deg_{r-1}(F_j),$$
$$\xi_{d_1+1}^j = \cdots = \xi_{d_2}^j = \deg_2(F_j) + \cdots + \deg_{r-1}(F_j),$$
$$\vdots$$
$$\xi_{d_{r-2}}^j = \cdots = \xi_{d_{r-1}}^j = \deg_{r-1}(F_j).$$

We denote by $Y(F_1), ..., Y(F_\ell)$ the Young diagrams determined by the ordered partition $e_1 + \cdots + e_{r-1} = n - e_r$ and the homogeneous degree of the K-polynomials $F_1(z)...F_\ell(z)$ shown in Fig. 2.2.1.

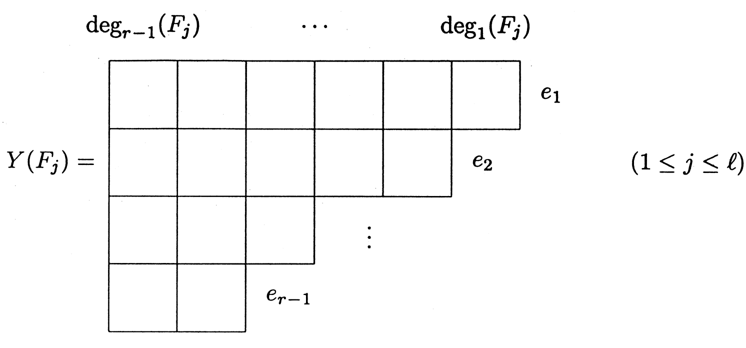

FIGURE 2.2.1

Note that the sequence $\{\xi_k^j\}_{1 \leqslant k \leqslant n-e_r}$ of positive integers corresponds to the young diagram $Y(F_j)$ $(1 \leqslant j \leqslant \ell)$. Similarly, we define the positive integers $\tilde{\xi}_k^j$ $(1 \leqslant j \leqslant \ell,\ 1 \leqslant k \leqslant n-e_1)$ as follows;

$$\begin{aligned}
\tilde{\xi}_1^j = \cdots = \tilde{\xi}_{d'_{r-1}}^j &= \deg_1(F_j) + \cdots + \deg_{r-1}(F_j) \\
\tilde{\xi}_{d'_{r-1}+1}^j = \cdots = \tilde{\xi}_{d'_{r-2}}^j &= \deg_2(F_j) + \cdots + \deg_{r-1}(F_j) \\
&\ \ \vdots \\
\tilde{\xi}_{d'_2+1}^j = \cdots = \tilde{\xi}_{d'_1}^j &= \deg_{r-1}(F_j).
\end{aligned}$$

Then the sequence $\{\tilde{\xi}_k^j\}_{1 \leqslant k \leqslant n-e_1}$ of positive integers corresponds to the Young diagram $\tilde{Y}(F_j)$ determined by the ordered partition $e_r + \cdots + e_2 = n - e_1$ and the homogeneous degrees of the K-polynomial $F_j(z)$ shown in Fig. 2.2.2.

For an $s = (s_1, \ldots, s_\ell) \in \mathbf{C}^\ell$, we define the linear combinations $\xi_k(s)$ and $\tilde{\xi}_k(s)$ of $s_1, \ldots, s_\ell$ as

$$\xi_k(s) = \xi_k^1 \cdot s_1 + \cdots + \xi_k^\ell \cdot s_\ell \qquad (1 \leqslant k \leqslant n - e_r)$$

and

$$\tilde{\xi}_k(s) = \tilde{\xi}_k^1 \cdot s_1 + \cdots + \tilde{\xi}_k^\ell \cdot s_\ell \qquad (1 \leqslant k \leqslant n - e_1).$$

For an integer a, we put $(a) = 1 - q^{-a}$; for example, $(2) = 1 - q^{-2}$.

Now the following is the main theorem of this paper, which will be proved in Section 3.

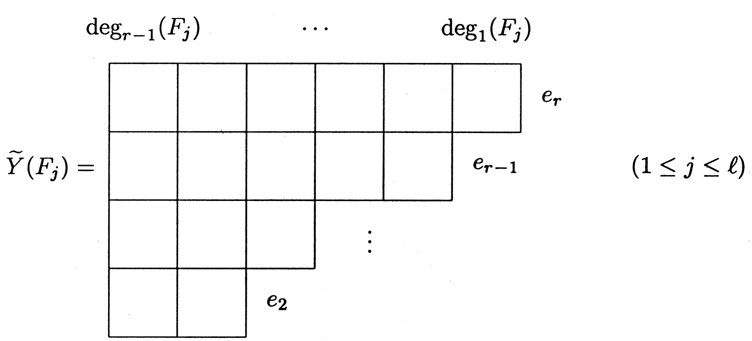

FIGURE 2.2.2

THEOREM 2.1. *For any* $\omega_s\chi = (\omega_{s_1}\chi_1, ..., \omega_{s_\ell}\chi_\ell) \in \Omega(K^\times)^\ell$, *we have*

$$\frac{Z_K(\omega_s\chi)}{\prod_{k=1}^{n-e_r}((n-k+1)/(1-q^{-\{\xi_k(s)+n-k+1\}}))}$$
$$= \frac{\tilde{Z}_K(\omega_s\chi)}{\prod_{k=1}^{n-e_1}((n-k+1)/(1-q^{-\{\tilde{\xi}_k(s)+n-k+1\}}))}.$$

Remark. If $r=2$, then (G, ρ, V) and $(\tilde{G}, \tilde{\rho}, \tilde{V})$ are the usual castling transforms of each other, as we noted in Subsection 2.1, and the identity in Theorem 2.1 becomes

$$\frac{Z_K(\omega_s\chi)}{\prod_{e_2<k\leqslant n}((k)/(1-q^{-\{(d,s)+k\}}))} = \frac{\tilde{Z}_K(\omega_s\chi)}{\prod_{e_1<k\leqslant n}((k)/(1-q^{-\{(d,s)+k\}}))},$$

where we denote by $\deg F_j$ the homogeneous degree of the K-polynomial $F_j(z)$ $(1 \leqslant j \leqslant \ell)$ and put $(d, s) = \deg F_1 \cdot s_1 + \cdots + \deg F_\ell \cdot s_\ell$. This is nothing but Igusa's result in [2].

3. PROOF OF MAIN THEOREM

3.1. *Measures*

We put

$$V'_K = \{x \in V_K \mid \operatorname{rank}(x) = n - e_r\}$$

and

$$\tilde{V}'_K = \{\tilde{x} \in \tilde{V}_K \mid \operatorname{rank}(\tilde{x}) = n - e_1\}.$$

The purpose of this section is to define measures on $\Delta^{-1}(\Delta(x))$ and $\tilde{\Delta}^{-1}(\tilde{\Delta}(\tilde{x}))$ for each $x \in V'_K$ and $\tilde{x} \in \tilde{V}'_K$, respectively.

The general linear group $GL_n(K)$ over K acts on V'_K by the usual matrix multiplication gx $(g \in GL_n(K),\ x \in V'_K)$. We define the action of $GL_n(K)$ on $\Delta(V'_K)$ by

$$g \cdot \Delta(x) = \Delta(gx) \qquad (g \in GL_n(K),\ x \in V'_K).$$

Then $GL_n(K)$ equivariantly acts on V'_K and $\Delta(V'_K)$ relative to Δ. The action of $GL_n(K)$ on V'_K is transitive, hence the action of $GL_n(K)$ on $\Delta(V'_K)$ is also transitive:

$$\Delta(V'_K) = GL_n(K) \cdot \Delta(x_0), \qquad x_0 = \begin{pmatrix} 1_{n-e_r} \\ 0 \end{pmatrix}.$$

Similarly, $GL_n^*(K)=\{g^*={}^t g^{-1} \mid g\in GL_n(K)\}$ equivariantly acts on $\tilde{V}'_K$ by

$$g^*\cdot\tilde{\Delta}(\tilde{x})=\tilde{\Delta}(g^*\tilde{x}) \qquad (g^*\in GL_n^*(K),\, \tilde{x}\in\tilde{V}'_K),$$

and $\tilde{\Delta}(\tilde{V}'_K)$ relative to $\tilde{\Delta}$ and the action of $GL_n^*(K)$ on $\tilde{\Delta}(\tilde{V}'_K)$ is transitive. We put

$$E_i=\begin{pmatrix} \varepsilon_i & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}\in GL_{e_i}(K),$$

in which $\varepsilon_i\in\{\pm 1\}$ is determined by

$$\det\begin{pmatrix} 0 & & E_i \\ & \cdots & \\ E_r & & 0 \end{pmatrix}=1 \qquad (2\leqslant i\leqslant r).$$

Then we have

$$\tilde{\Delta}(\tilde{V}'_K)=GL_n^*(K)\cdot\tilde{\Delta}(\tilde{x}_0), \qquad \tilde{x}_0=\left(\begin{array}{ccc} & 0 & \\ \hline 0 & & E_2 \\ & \cdots & \\ E_r & & 0 \end{array}\right).$$

Remark. Since $\Delta(x_0)=\tilde{\Delta}(\tilde{x}_0)$, we have $\Delta(V'_K)=GL_n(K)\cdot\Delta(x_0)=GL_n^*(K)\cdot\tilde{\Delta}(\tilde{x}_0)=\tilde{\Delta}(\tilde{V}'_K)$.

Let H_0 and $\tilde{H}_0$ be the isotropy subgroups of $GL_n(K)$ and $GL_n^*(K)$ at $\Delta(x_0)$ and $\tilde{\Delta}(\tilde{x}_0)$, respectively. Then we can identify

$$\Delta(V'_K)=GL_n(K)/H_0$$

and

$$\tilde{\Delta}(\tilde{V}'_K)=GL_n^*(K)/\tilde{H}_0.$$

We denote by $SP_{e_1,\ldots,e_{r-1}}$ the subgroup of $P_{e_1,\ldots,e_{r-1}}$ consisting of all matrices $P=(p_{ij})\in P_{e_1,\ldots,e_{r-1}}$ satisfying $p_i\in SL_{e_i}(\bar{K})$ $(1\leqslant i\leqslant r-1)$ and put $SP_{e_1,\ldots,e_{r-1}}(K)=SP_{e_1,\ldots,e_{r-1}}\cap GL_{n-e_r}(K)$. Similarly, we define $SP_{e_r,\ldots,e_2}$ and $SP_{e_r,\ldots,e_2}(K)$. We can show that, for any $x\in V'_K$ and $\tilde{x}\in\tilde{V}'_K$,

$$\Delta^{-1}(\Delta(x))=x\cdot SP_{e_1,\ldots,e_{r-1}}^{-1}$$

and

$$\tilde{\Delta}^{-1}(\tilde{\Delta}(\tilde{x})) = \tilde{x} \cdot SP^{-1}_{e_r, \ldots, e_2};$$

hence we have that H_0 consists of all matrices of the form

$$\begin{pmatrix} p_1 & \cdots & p_{ij} \\ & \ddots & \vdots \\ 0 & & p_r \end{pmatrix},$$

where $p_i \in SL_{e_j}(K)$ $(1 \leqslant i \leqslant r-1)$, $p_r \in GL_{e_r}(K)$, and $p_{ij} \in M_{e_i, e_j}(K)$ $(1 \leqslant i < j \leqslant r)$ and $\tilde{H}_0$ consists of all matrices of the form

$$\begin{pmatrix} \tilde{p}_1 & & 0 \\ \vdots & \ddots & \\ \tilde{p}_{ji} & \cdots & \tilde{p}_r \end{pmatrix},$$

where $\tilde{p}_1 \in GL_{e_1}(K)$, $\tilde{p}_i \in SL_{e_i}(K)$ $(2 \leqslant j \leqslant r)$, and $\tilde{p}_{ji} \in M_{e_j, e_i}(K)$ $(1 \leqslant i < j \leqslant r)$.

We put

$$d_{H_0}(h) = \frac{\prod_{i=1}^{r} |dp_i|_K \prod_{1 \leqslant i < j < \leqslant r} |dp_{ij}|_K}{|\det(h)|_K^{e_r}} \qquad \left(h = \begin{pmatrix} p_1 & \cdots & p_{ij} \\ & \ddots & \vdots \\ 0 & & p_r \end{pmatrix} \in H_0 \right)$$

and

$$d_{\tilde{H}_0}(\tilde{h}) = \frac{\prod_{i=1}^{r} |d\tilde{p}_i|_K \prod_{1 \leqslant i < j < \leqslant r} |d\tilde{p}_{ji}|_K}{|\det(\tilde{h})|_K^{e_r}} \qquad \left(\tilde{h} = \begin{pmatrix} \tilde{p}_1 & & 0 \\ & \ddots & \\ \tilde{p}_{ji} & & \tilde{p}_r \end{pmatrix} \in \tilde{H}_0 \right).$$

Then d_{H_0} and $d_{\tilde{H}_0}$ define the Haar measures on H_0 and $\tilde{H}_0$, respectively. Furthermore, they satisfy

$$d_{H_0}(hh_0) = |\det(h_0)|_K^{n-e_r}\, d_{H_0}(h) \qquad (h \in H_0)$$

and

$$d_{\tilde{H}_0}(\tilde{h}\tilde{h}_0) = |\det(\tilde{h}_0)|_K^{n-e_1}\, d_{\tilde{H}_0}(\tilde{h}) \qquad (\tilde{h} \in \tilde{H}_0);$$

hence the modules Δ_{H_0} and $\Delta_{\tilde{H}_0}$ of H_0 and $\tilde{H}_0$, respectively, are given by

$$\Delta_{H_0}(h) = |\det(h)|_K^{n-e_r} \qquad (h \in H_0)$$

and

$$\Delta_{\tilde{H}_0}(\tilde{h}) = |\det(\tilde{h})|_K^{n-e_1} \qquad (\tilde{h} \in \tilde{H}_0).$$

By the theorem of Weil (cf. [8, Chap. II, Sect. 9]), we have the following lemma.

LEMMA 3.1. (1) *There exists a measure $d\mu_K$ on $\Delta(V'_K)$, unique up to a constant factor, satisfying*

$$d\mu_K(g \cdot z) = |\det(g)|_K^{n-e_r}\, d\mu_k(z) \qquad (g \in GL_n(K)).$$

(2) *There exists a measure $d\tilde{\mu}_K$ on $\tilde{\Delta}(\tilde{V}'_K)$, unique up to a constant factor, satisfying*

$$d\tilde{\mu}_K(g^* \cdot \tilde{z}) = |\det(g^*)|_K^{n-e_1}\, d\tilde{\mu}_K(\tilde{z}) \qquad (g^* \in GL_n^*(K)).$$

For each i $(1 \leqslant i \leqslant r-1)$, we put $\lambda_0(i) = (1, ..., d_i)$. Then we have $\lambda_0(i)^* = (d_i + 1, ..., n)$ and $\mathrm{sgn}(\lambda_0(i)) = 1$ (see Subsection 2.1). Let V_K° be the open subsets of V'_K and $\tilde{V}'_K$, respectively, defined by

$$V_K^\circ = \{x \in V'_K \mid \det_{\lambda_0(i)}(x^{d_i}) \neq 0 (1 \leqslant i \leqslant r-1)\}$$

and

$$\tilde{V}_K^\circ = \{\tilde{x} \in \tilde{V}'_K \mid \det_{\lambda_0(i)^*}(\tilde{x}^{d'_i}) \neq 0 (1 \leqslant i \leqslant r-1)\}.$$

The stabilizer G_K° of $\Delta(\overset{\circ}{V}_K)$ in $GL_n(K)$ consists of all matrices of the form

$$\begin{pmatrix} g_1 & & 0 \\ \vdots & \ddots & \\ g_{ji} & \cdots & g_r \end{pmatrix},$$

where $g_i \in GL_{e_i}(K)$ $(1 \leqslant i \leqslant r)$ and $g_{ji} \in M_{e_j, e_i}(K)$ $(1 \leqslant i < j \leqslant r)$, and the stabilizer $\tilde{G}_K^\circ$ of $\tilde{\Delta}(\tilde{V}_K^\circ)$ in $GL_n^*(K)$ consists of all matrices of the form

$$\begin{pmatrix} \tilde{g}_1 & \cdots & \tilde{g}_{ij} \\ & \ddots & \vdots \\ 0 & & \tilde{g}_r \end{pmatrix},$$

where $\tilde{g}_i \in GL_{e_i}(K)$ $(1 \leqslant i \leqslant r)$ and $\tilde{g}_{ij} \in M_{e_i, e_j}(K)$ $(1 \leqslant i < j \leqslant r)$. Furthermore, we can show that G_K° and $\tilde{G}_K^\circ$ are transitive on $\Delta(V_K^\circ)$ and $\tilde{\Delta}(\tilde{V}_K^\circ)$, respectively,

$$\Delta(V_K^\circ) = G_K^\circ \cdot \Delta(x_0)$$

and

$$\tilde{\Delta}(\tilde{V}_K^\circ) = \tilde{G}_K^\circ \cdot \tilde{\Delta}(\tilde{x}_0).$$

Every $x \in V_K^\circ$ can be expressed as

$$x = \begin{pmatrix} 1_{e_1} & & 0 \\ & \ddots & \\ & & 1_{e_{r-1}} \\ & y_{ji} & \end{pmatrix} \cdot \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_{r-1} \end{pmatrix},$$

where $y_{ji} \in M_{e_j, e_i}(K)$ $(1 \leqslant i < j \leqslant r)$ and $p_i \in GL_{e_i}(K)$ $(1 \leqslant i \leqslant r-1)$. We put, for each i $(1 \leqslant i \leqslant r-1)$,

$$t_i = \det \begin{pmatrix} p_1 & & 0 \\ & \ddots & \\ 0 & & p_i \end{pmatrix} \neq 0, \qquad y_i = {}^t({}^t y_{i+1\, i}, ..., {}^t y_{ri}) \in M_{d_i', e_i}(K).$$

Then, for any $\lambda \in \Lambda_i$, $\det_\lambda(x^{d_i})$ is the product of t_i and a polynomial in the entries of y_i. Hence, $t = (t_1, ..., t_{r-1})$ and $y = (y_1, ..., y_{r-1})$ form local coordinates on $\Delta(V_K^\circ)$. Similarly, every $\tilde{x} \in \tilde{V}_K^\circ$ can be expressed as

$$\tilde{x} = \begin{pmatrix} & \tilde{y}_{ij} & \\ & & 1_{e_2} \\ & \cdots & \\ 1_{e_r} & & 0 \end{pmatrix} \cdot \begin{pmatrix} \tilde{p}_r & & 0 \\ & \ddots & \\ 0 & & \tilde{p}_2 \end{pmatrix},$$

where $\tilde{y}_{ij} \in M_{e_i, e_j}(K)$ $(1 \leqslant i < j \leqslant r)$ and $\tilde{p}_i \in GL_{e_i}(K)$ $(2 \leqslant i \leqslant r)$. We put, for each i $(2 \leqslant i \leqslant r)$,

$$\tilde{t}_i = \det \begin{pmatrix} \tilde{p}_r & & 0 \\ & \ddots & \\ 0 & & \tilde{p}_i \end{pmatrix} \neq 0, \qquad \tilde{y}_i = {}^t({}^t\tilde{y}_{1i}, ..., {}^t\tilde{y}_{i-1\, i}) \in M_{d_{i-1}, e_i}(K).$$

Then $\tilde{t} = (\tilde{t}_r, ..., \tilde{t}_2)$ and $\tilde{y} = (\tilde{y}_r, ..., \tilde{y}_2)$ form local coordinates on $\tilde{\Delta}(\tilde{V}_K^\circ)$.

LEMMA 3.2. *We can normalize the measures $d\mu_K$ and $d\tilde{\mu}_K$ as*

$$d\mu_K(t, y) = \prod_{i=1}^{r-1} |t_i|_K^{e_i + e_{i+1} - 1} \, |dt \wedge dy|_K \qquad \textit{on } \Delta(V_K^\circ)$$

and

$$d\tilde{\mu}_K(\tilde{t}, \tilde{y}) = \prod_{i=2}^{r} |\tilde{t}_i|_K^{e_{i-1}+e_i-1} \, |d\tilde{t} \wedge d\tilde{y}|_K \qquad \textit{on } \tilde{\Delta}(\tilde{V}_K^\circ).$$

Proof. We consider the relative invariance of $d\mu_K$ on $\Delta(V_K^\circ)$ under the action of the stabilizer G_K° of $\Delta(V_K^\circ)$; by Lemma 3.1, we have, for

$$g = \begin{pmatrix} g_1 & & 0 \\ \vdots & \ddots & \\ g_{ji} & \cdots & g_r \end{pmatrix} \in G_K^\circ,$$

$$d\mu_K(g \cdot z) = \prod_{i=1}^{r} |\det(g_i)|_K^{n-e_r} \, d\mu_K(z). \tag{3.1}$$

For the local coordinates $t = (t_1, ..., t_{r-1})$ and y on $\Delta(V_K^\circ)$, there exists a polynomial $\psi(t, y)$ in $t_i^{\pm 1}$ $(1 \leqslant i \leqslant r-1)$ and the entries of y such that

$$d\mu_K(t, y) = |\psi(t, y) \, dt \wedge dy|_K \qquad \text{on } \Delta(V_K^\circ).$$

If we take

$$g_1 = \begin{pmatrix} 1_{e_1} & 0 \\ y_1' & 1_{n-e_1} \end{pmatrix} \in G_K^\circ \qquad (y_1' \in M_{n-e_1, e_1}(K)),$$

then we have

$$d\mu_K(g_1 \cdot z) = |\psi(t, y_1 + y_1', y_2, ..., y_{r-1}) \, dt \wedge dy|_K.$$

On the other hand, by the above identity (3.1), $d\mu_K$ is invariant under the action of $g_1 \in G_K^\circ$. Hence, we have

$$|\psi(t, y_1 + y_1', y_2, ..., y_{r-1}) \, dt \wedge dy|_K = |\psi(t, y_1, y_2, ..., y_{r-1}) \, dt \wedge dy|_K.$$

This implies that $\psi(t, y)$ is independent of y_1. In the same way, we can show that $\psi(t, y)$ is independent of y_i $(2 \leqslant i \leqslant r-1)$. Hence $\psi(t, y)$ is independent of y. If we take

$$g_2 = \begin{pmatrix} 1_{e_1} & & & 0 \\ & \ddots & & \\ & & g_{r-1} & \\ 0 & & & 1_{e_r} \end{pmatrix} \in G_K^\circ \qquad (g_{r-1} \in GL_{e_{r-1}}(K)),$$

then, by (3.1),

$$d\mu_K(g_2 \cdot z) = |c|_K^{n-e_r}\, d\mu_K(z) \qquad (c = \det(g_2) \neq 0).$$

On the other hand, we have

$$d\mu_K(g_2 \cdot z) = |c^{n-e_r-(e_{r-1}+e_r)+1}\psi(t_1, ..., t_{r-2}, ct_{r-1}, y)\, dt \wedge dy|_K.$$

Hence, we have

$$|\psi(t_1, ..., t_{r-2}, ct_{r-1}, y)\, dt \wedge dy|_K = |c^{e_{r-1}+e_r-1}\psi(t_1, ..., t_{r-1}, y)\, dt \wedge dy|_K.$$

This implies hat $\psi(t, y)$ is homogeneous of degree $e_{r-1}+e_r-1$ with respect to t_{r-1}. Furthermore, we have $\psi(t, y)$ is homogeneous of degree $e_{r-i}+e_{r-i+1}-1$ with respect to t_{r-i} $(1 \leqslant i \leqslant r-1)$; we can prove it by an induction on i. Therefore, we have

$$\psi(t, y) = \alpha \prod_{i=1}^{r-1} t_i^{e_i+e_{i+1}-1}$$

with some constant $\alpha \neq 0$. Since $d\mu_K$ is unique up to a nonzero constant factor, we can normalize $d\mu_K$ so that $\alpha = 1$. Hence, we have

$$d\mu_K(t, y) = \prod_{i=1}^{r-1} |t_i|_K^{e_i+e_{i+1}-1}\, |dt \wedge dy|_K \qquad \text{on } \Delta(V_K^\circ).$$

In the same way, by the relative invariance of $d\tilde{\mu}_K$ on $\tilde{\Delta}(\tilde{V}_K^\circ)$ under the action of the stabilizer $\tilde{G}_K^\circ$ of $\tilde{\Delta}(\tilde{V}_K^\circ)$, we can show that

$$d\tilde{\mu}_K(\tilde{t}, \tilde{y}) = \prod_{i=2}^{r} |\tilde{t}_i|_K^{e_{i-1}+e_i-1}\, |d\tilde{t} \wedge d\tilde{y}|_K \qquad \text{on } \tilde{\Delta}(\tilde{V}_K^\circ).$$

We have thus proved our lemma. ▮

We define the mapping $\iota^* = G_K^\circ \to \tilde{G}_K^\circ$ by

$$\iota^*(g) = g^* \begin{pmatrix} \det(g)^{-1} & & & & 0 \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 0 & & & & \det(g) \end{pmatrix} \qquad (g \in G_K^\circ)$$

and the mapping $\iota\colon \Delta(V_K^\circ) = G_K^\circ \cdot \Delta(x_0) \to \tilde{\Delta}(\tilde{V}_K^\circ) = \tilde{G}_K^\circ \cdot \tilde{\Delta}(\tilde{x}_0)$ by

$$\iota(g \cdot \Delta(x_0)) = \iota^*(g) \cdot \tilde{\Delta}(\tilde{x}_0).$$

Then the mapping ι gives a homeomorphism. Let $t=(t_1, ..., t_{r-1})$ and y be the local coordinates on a neighborhood of $\Delta(x_0)$ in $\Delta(V_K^\circ)$ and $\tilde{t}=(\tilde{t}_r, ..., \tilde{t}_2)$ and $\tilde{y}$ the local coordinates on a neighborhood of $\tilde{\Delta}(\tilde{x}_0)=\iota(\Delta(x))$ in $\tilde{\Delta}(\tilde{V}_K^\circ)$. Then we have

$$\prod_{i=2}^{r} |\tilde{t}_i|_K^{e_{i-1}+e_i-1} |d\tilde{t} \wedge d\tilde{y}|_K = \prod_{i=1}^{r-1} |\tilde{t}_i|_K^{e_i+e_{i+1}-1} |dt \wedge dy|_K.$$

Hence, we can choose a neighborhood U of $\Delta(x_0)=\tilde{\Delta}(\tilde{x}_0)$ in $\Delta(V'_K)=\tilde{\Delta}(\tilde{V}'_K)$ such that

$$U \subset \Delta(V_K^\circ) \cap \tilde{\Delta}(\tilde{V}_K^\circ), \qquad \iota\colon U \to U$$

and

$$d\tilde{\mu}_K(\iota(z)) = d\mu_K(z) \tag{3.2}$$

on U. On the one hand, $\Delta(V'_K)$ and $\tilde{\Delta}(\tilde{V}'_K)$ are covered with some open subsets which are isomorphic to $\Delta(V_K^\circ)$ and $\tilde{\Delta}(\tilde{V}_K^\circ)$, respectively. Therefore, by (3.2), we have the following lemma.

Lemma 3.3. *For any integrable function φ on $\Delta(V'_K)=\tilde{\Delta}(\tilde{V}'_K)$, we have*

$$\int_{z \in \Delta(V'_K)} \varphi(z)\, d\mu_K(z) = \int_{\tilde{z} \in \tilde{\Delta}(\tilde{V}'_K)} \varphi(\tilde{z})\, d\tilde{\mu}_K(\tilde{z}).$$

For every $x \in V'_K$, we define the measure $\theta_{\Delta(x)}(x)$ on $\Delta^{-1}(\Delta(x))$ by

$$\theta_{\Delta(x)} = [dx/d\mu_K(\Delta(x))]_{\Delta^{-1}(\Delta(x))}$$

(cf. [9, Chap. I, Sect. 5]). Since dx and $d\mu_K$ have same relative invariance under the action of $GL_n(K)$; $d(gx)=|\det(g)|_K^{n-e_r}\, dx$, $d\mu_K(g\cdot z)=|\det(g)|_K^{n-e_r}\, d\mu_K(z)$ $(g \in GL_n(K))$ (see Lemma 3.1), $\theta_{\Delta(x)}$ is $GL_n(K)$-invariant:

$$\theta_{g\cdot\Delta(x)}(gx) = \theta_{\Delta(x)}(x) \qquad (g \in GL_n(K),\, x \in V'_K).$$

We normalize the measure $\theta_{\Delta(x)}$ as follows: for every bounded continuous function ϕ on (V_K) and every integrable function Φ on V_K,

$$\int_{x \in V_K} \phi(\Delta(x))\, \Phi(x)\, dx = \int_{z \in \Delta(V'_K)} \phi(z)\, d\mu_K(z) \int_{x \in \Delta^{-1}(z)} \Phi(x)\, \theta_z(x). \tag{3.3}$$

Similarly, for every $\tilde{x} \in \tilde{V}'_K$, we define the $GL_n^*(K)$-invariant measure $\tilde{\theta}_{\tilde{\Delta}(\tilde{x})}$ on $\tilde{\Delta}^{-1}(\tilde{\Delta}(\tilde{x}))$ by

$$\tilde{\theta}_{\tilde{\Delta}(\tilde{x})}(\tilde{x}) = [\, d\tilde{x}/d\tilde{\mu}_K(\tilde{\Delta}(\tilde{x}))\,]_{\tilde{\Delta}^{-1}(\tilde{\Delta}(\tilde{x}))}.$$

We normalize the measure $\tilde{\theta}_{\tilde{\Delta}(\tilde{x})}$ as follows; for every bounded continuous function $\tilde{\phi}$ on $\tilde{\Delta}(\tilde{V}_K)$ and every integrable function $\tilde{\Phi}$ on $\tilde{V}_K$,

$$\int_{\tilde{x} \in \tilde{V}_K} \tilde{\phi}(\tilde{\Delta}(\tilde{x}))\, \tilde{\Phi}(\tilde{x})\, d\tilde{x} = \int_{\tilde{z} \in \tilde{\Delta}(\tilde{V}'_K)} \tilde{\phi}(\tilde{z})\, d\tilde{\mu}_K(\tilde{z}) \int_{\tilde{x} \in \tilde{\Delta}^{-1}(\tilde{z})} \tilde{\Phi}(\tilde{x})\, \tilde{\theta}_{\tilde{z}}(\tilde{x}). \tag{3.4}$$

3.2. *Lemma for $\wp$-adic Integrals*

We put $V_K(\mathcal{O}_K) = M_{n,\, n-e_r}(\mathcal{O}_K)$ and $\tilde{V}_K(\mathcal{O}_K) = M_{n,\, n-e_1}(\mathcal{O}_K)$. Let ϕ_0 and $\tilde{\phi}_0$ be continuous functions on $\Delta(V_K(\mathcal{O}_K))$ and $\tilde{\Delta}(\tilde{V}_K(\mathcal{O}_K))$, respectively. In this section, we give a lemma for the $\wp$-adic integrals $\int_{x \in V_K(\mathcal{O}_K)} \phi_0(\Delta(x))\, dx$ and $\int_{\tilde{x} \in \tilde{V}_K(\mathcal{O}_K)} \tilde{\phi}_0(\tilde{\Delta}(\tilde{x}))\, d\tilde{x}$.

For $m \geqslant 1$, we put

$$U_K(m) = \mathcal{O}_K^m - \pi_K \mathcal{O}_K^m.$$

We define the open compact subsets U_0 and $\tilde{U}_0$ of $K^{m_1} \oplus \cdots \oplus K^{m_{r-1}}$ by

$$U_0 = \{ z = (z^i) \in \Delta(V_K(\mathcal{O}_K)) \mid z^i \in U_K(m_i) (1 \leqslant i \leqslant r-1) \}$$

and

$$\tilde{U}_0 = \{ \tilde{z} = (\tilde{z}^i) \in \tilde{\Delta}(\tilde{V}_K(\mathcal{O}_K)) \mid \tilde{z}^i \in U_K(m_i) (1 \leqslant i \leqslant r-1) \}.$$

For $k = (k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$, we put

$$U_k = \{ z = (z^i) \in \Delta(V_K(\mathcal{O}_K)) \mid z^i \in \pi_K^{k_i} U_K(m_i) (1 \leqslant i \leqslant r-1) \}.$$

Then we have

$$\Delta(V'_K) = \coprod_{k \in \mathbf{Z}^{r-1}} U_k \qquad \text{(disjoint union)}.$$

Hence, in view of

$$d\mu_K(\pi_K^k z) = \prod_{i=1}^{r-1} q^{-k_i(e_i + e_{i+1})}\, d\mu_K(z) \qquad (k = (k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}),$$

(3.3) becomes

$$\int_{x \in V_K} \phi(\Delta(x))\, \Phi(x)\, dx$$
$$= \sum_{k \in \mathbf{Z}^{r-1}} \prod_{i=1}^{r-1} q^{-k_i(e_i+e_{i+1})} \int_{z \in U_0} \phi(\pi_K^k z)\, d\mu_K(z) \int_{x \in \Delta^{-1}(\pi_K^k z)} \Phi(x)\, \theta_{\pi_K^k z}(x). \tag{3.5}$$

In the same way, (3.4) becomes

$$\int_{\tilde{x} \in \tilde{V}} \tilde{\phi}(\tilde{\Delta}(\tilde{x}))\, \tilde{\Phi}(\tilde{x})\, d\tilde{x}$$
$$= \sum_{k \in \mathbf{Z}^{r-1}} \prod_{i=2}^{r} q^{-k_{i-1}(e_{i-1}+e_i)} \int_{\tilde{z} \in \tilde{U}_0} \tilde{\phi}(\pi_K^k \tilde{z})\, d\tilde{\mu}_K(\tilde{z}) \int_{\tilde{x} \in \tilde{\Delta}^{-1}(\pi_K^k \tilde{z})} \tilde{\Phi}(\tilde{x})\, \tilde{\theta}_{\pi_K^k \tilde{z}}(\tilde{x}). \tag{3.6}$$

For $k = (k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$, we put

$$I_k(z) = \int_{x \in \Delta^{-1}(\pi_K^k z) \,\cap\, V_K(\mathcal{O}_K)} \theta_{\pi_K^k z}(x) \qquad (z \in U_0)$$

and

$$\tilde{I}_k(z) = \int_{\tilde{x} \in \tilde{\Delta}^{-1}(\pi_K^k \tilde{z}) \cap \tilde{V}_K(\mathcal{O}_K)} \tilde{\theta}_{\pi_K^k \tilde{z}(\tilde{x})} \qquad (\tilde{z} \in \tilde{U}_0).$$

In (3.5), we take the constant function 1 as ϕ and the characteristic function $Ch_{\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K)}$ of $\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K)$ as Φ. Then we have

$$\mathrm{vol}(\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K)) = \prod_{i=1}^{r-1} q^{-k_i(e_i+e_{i+1})} \int_{z \in U_0} I_k(z)\, d\mu_K(z). \tag{3.7}$$

In the same way, from (3.6), we have

$$\mathrm{vol}(\tilde{\Delta}^{-1}(\tilde{U}_k) \cap \tilde{V}_K(\mathcal{O}_K)) = \prod_{i=2}^{r} q^{-k_i(e_i+e_{i-1})} \int_{\tilde{z} \in \tilde{U}_0} \tilde{I}_k(\tilde{z})\, d\tilde{\mu}_K(\tilde{z}). \tag{3.8}$$

Since $\theta_{\Delta(x)}$ is $GL_n(K)$-invariant (see Subsection 3.2), we have $I_k(g \cdot z) = I_k(z)$ $(g \in GL_n(K))$. Similarly, we have $\tilde{I}_k(g^* \cdot \tilde{z}) = \tilde{I}_k(\tilde{z})$ $(g^* \in GL_n^*(K))$. On the other hand, U_0 is a $GL_n(\mathcal{O}_K)$-homogeneous space and $\tilde{U}_0$ is also a $GL_n^*(\mathcal{O}_K)$-homogeneous space; $U_0 = GL_n(\mathcal{O}_K) \cdot \Delta(x_0)$ and $\tilde{U}_0 = GL_n^*(\mathcal{O}_K) \cdot \tilde{\Delta}(\tilde{x}_0)$. Hence, $I_k(z)$ and $\tilde{I}_k(\tilde{z})$ are independent of $z \in U_0$ and $\tilde{z} \in \tilde{U}_0$ respectively, and we write $I_k = I_k(z)$ and $\tilde{I}_k = \tilde{I}_k(\tilde{z})$. We denote by $\mu_K(U_0)$ and

$\tilde{\mu}_K(\tilde{U}_0)$ the volumes of U_0 and $\tilde{U}_0$ with respect to the measures $d\mu_K$ and $d\tilde{\mu}$, respectively,

$$\mu_K(U_0) = \int_{z \in U_0} d\mu_K(z)$$

and

$$\tilde{\mu}_K(\tilde{U}_0) = \int_{\tilde{z} \in \tilde{U}_0} d\tilde{\mu}_K(\tilde{z}).$$

Then, from (3.7) and (3.8), we have

$$I_k = \frac{1}{\mu_K(U_0)} \prod_{i=1}^{r-1} q^{-k_i(e_i + e_{i+1})} \operatorname{vol}(\varDelta^{-1}(U_k) \cap V_K(\mathcal{O}_K)) \tag{3.9}$$

and

$$\tilde{I}_k = \frac{1}{\tilde{\mu}_K(\tilde{U}_0)} \prod_{i=2}^{r} q^{-k_{i-1}(e_{i-1} + e_i)} \operatorname{vol}(\tilde{\varDelta}^{-1}(\tilde{U}_k) \cap \tilde{V}_K(\mathcal{O}_K)). \tag{3.10}$$

Now we consider the computation of $\operatorname{vol}(\varDelta^{-1}(U_k) \cap V_K(\mathcal{O}_K))$ and $\operatorname{vol}(\tilde{\varDelta}^{-1}(\tilde{U}_k) \cap \tilde{V}_K(\mathcal{O}_K))$.

For an $x \in V'_K \cap V_K(\mathcal{O}_K)$, we denote by $\operatorname{ord}_K(x)_i$ the minimum order of all d_i-minors of x^{d_i} $(1 \leqslant i \leqslant r-1)$, then we have $0 \leqslant \operatorname{ord}_K(x)_1 \leqslant \cdots \leqslant \operatorname{ord}_K(x)_{r-1}$. Hence, unless $k = (k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ satisfies $0 \leqslant k_1 \leqslant \cdots \leqslant k_{r-1}$, then $\varDelta^{-1}(U_k) \cap V_K(\mathcal{O}_K)$ is empty and $\operatorname{vol}(\varDelta^{-1}(U_k) \cap V_K(\mathcal{O}_K)) = 0$. Similarly, unless $k = (k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ satisfies $0 \leqslant k_{r-1} \leqslant \cdots \leqslant k_1$, then $\operatorname{vol}(\tilde{\varDelta}^{-1}(\tilde{U}_K) \cap \tilde{V}_K(\mathcal{O}_K)) = 0$. Hence we have the following lemma.

LEMMA 3.4. (1) *If $k = (k_1, ..., k_{r-1})$ is not an increasing sequence of non-negative integers, then we have $I_k = 0$.*

(2) *If $k = (k_1, ..., k_{r-1})$ is not a decreasing sequence of non-negative integers, then we have $\tilde{I}_k = 0$.*

For a sequence $h = (h_1, ..., h_{n-e_r})$ of non-negative integers, we denote by $\operatorname{Trig}(h)$ the subset of $V_K(\mathcal{O}_K)$ consisting of all matrices of the form

$$\begin{pmatrix} \begin{matrix} \pi_K^{h_1} & & * \\ & \ddots & \\ 0 & & \pi_K^{h_{n-e_r}} \end{matrix} \\ \hline 0 \end{pmatrix},$$

and we define the open subset $\mathscr{F}(h)$ of $V_K(\mathcal{O}_K)$ by

$$\mathscr{F}(h) = GL_n(\mathcal{O}_K)\,\mathrm{Trig}(h).$$

Then we have

$$\mathrm{vol}(\mathscr{F}(h)) = \prod_{i=1}^{n-e_r} (n-i+1)\, q^{-(n-i+1)\, h_i}.$$

This can be proved by an induction on $n-e_r$ (cf. [2, Sect. 8]).

For $k=(k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ satisfying $0 \leqslant k_1 \leqslant \cdots \leqslant k_{r-1}$, we have

$$\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K) = \coprod_{\substack{h_1, ..., h_{n-e_r} \geqslant 0 \\ h_1 + \cdots + h_{d_i} = k_i (1 \leqslant i \leqslant r-1)}} \mathscr{F}(h),$$

hence

$$\mathrm{vol}(\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K)) = \sum_{\substack{h_1, ..., h_{n-e_r} \geqslant 0 \\ h_1 + \cdots + h_{d_i} = k_i (1 \leqslant i \leqslant r-1)}} \prod_{i=1}^{n-e_r} (n-i+1)\, q^{-(n-i+1)\, h_i}.$$

Similarly, for $k=(k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ satisfying $0 \leqslant k_{r-1} \leqslant \cdots \leqslant k_1$, we have

$$\mathrm{vol}(\tilde{\Delta}^{-1}(\tilde{U}_k) \cap \tilde{V}_K(\mathcal{O}_K)) = \sum_{\substack{h_1, ..., h_{n-e_1} \geqslant 0 \\ h_1 + \cdots + h_{d_{r-i}} = k_i (1 \leqslant i \leqslant r-1)}} \prod_{i=1}^{n-e_1} (n-i+1)\, q^{-(n-i+1)\, h_i}.$$

Thus we have the explicit formulae $\mathrm{vol}(\Delta^{-1}(U_k) \cap V_K(\mathcal{O}_K))$ and $\mathrm{vol}(\tilde{\Delta}^{-1}(\tilde{U}_k) \cap \tilde{V}_K(\mathcal{O}_K))$. We put them into (3.9) and (3.10), respectively; then we have the following lemma.

Lemma 3.5. *For $k=(k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ satisfying $0 \leqslant k_1 \leqslant \cdots \leqslant k_{r-1}$, we have*

$$I_k = \frac{1}{\mu_K(U_0)} \prod_{i=1}^{r-1} q^{-k_i(e_i + e_{i+1})} \sum_{\substack{h_1, ..., h_{n-e_r} \geqslant 0 \\ h_1 + \cdots + h_{d_i} = k_i (1 \leqslant i \leqslant r-1)}} \times \prod_{j=1}^{n-e_r} (n-j+1)\, q^{-(n-j+1)\, h_j}$$

and, for $k=(k_1, ..., k_{r-1}) \in \mathbf{Z}^{r-1}$ *satisfying* $0 \leqslant k_{r-1} \leqslant \cdots \leqslant k_1$, *we have*

$$\tilde{I}_k = \frac{1}{\tilde{\mu}_K(\tilde{U}_0)} \prod_{i=2}^{r} q^{-k_{i-1}(e_{i-1}+e_i)} \sum_{\substack{h_1, ..., h_{n-e_1} \geqslant 0 \\ h_1 + \cdots + h_{d'_{r-1}} = k_i (1 \leqslant i \leqslant r-1)}} \times \prod_{j=1}^{n-e_1} (n-j+1)\, q^{-(n-j+1)\, h_j}.$$

In (3.5) and (3.6), we take the characteristic functions $\mathrm{Ch}_{V_K(\mathcal{O}_K)}$ and $\mathrm{Ch}_{\tilde{V}_K(\mathcal{O}_K)}$ of $V_K(\mathcal{O}_K)$ and $\tilde{V}_K(\mathcal{O}_K)$ as Φ and $\tilde{\Phi}$, respectively. Then, by Lemma 3.4, we have the following lemma for $\wp$-adic integrals.

Lemma 3.6. *Let* ϕ_0 *and* $\tilde{\phi}_0$ *be continuous functions on* $\Delta(V_K(\mathcal{O}_K))$ *and* $\tilde{\Delta}(\tilde{V}_K(\mathcal{O}_K))$, *respectively. Then we have*

$$\int_{x \in V_K(\mathcal{O}_K)} \phi_0(\Delta(x))\, dx = \frac{1}{\mu_K(U_0)} \sum_{\substack{h_i \geqslant 0 \\ (1 \leqslant i \leqslant n-e_r)}} \prod_{i=1}^{n-e_r} (n-i+1)\, q^{-(n-i+1)\, h_i} \times \int_{z \in U_0} \phi_0(\pi_K^{h_1 + \cdots + h_{d_i}} z_i)\, d\mu_K(z)$$

and

$$\int_{\tilde{z} \in \tilde{V}(\mathcal{O}_K)} \tilde{\phi}_0(\tilde{\Delta}(\tilde{x}))\, d\tilde{x} = \frac{1}{\tilde{\mu}_K(\tilde{U}_0)} \sum_{\substack{h_i \geqslant 0 \\ (1 \leqslant i \leqslant n-e_1)}} \prod_{i=1}^{n-e_1} (n-i+1)\, q^{-(n-i+1)\, h_i} \times \int_{\tilde{z} \in \tilde{U}_0} \tilde{\phi}_0(\pi_K^{h_1 + \cdots + h_{d'_{r-i}}} \tilde{z}_i)\, d\tilde{\mu}_K(\tilde{z}).$$

3.3. *Proof of the Main Theorem*

Let us prove our main theorem—Theorem 2.1.

In Lemma 3.6, if we take $\omega(F)$ as ϕ_0, then we have

$$Z_K(\omega)=\frac{1}{\mu_K(U_0)}\sum_{\substack{h_i\geqslant 0\\(1\leqslant i\leqslant n-e_r)}}\prod_{i=1}^{n-e_r}(n-i+1)\,q^{-(n-i+1)\,h_i}$$

$$\times\int_{z\in U_0}\omega(F(\pi_K^{h_1+\cdots+h_{d_i}}z_i))\,d\mu_K(z).$$

We put

$$E_K(\omega)=\frac{1}{\mu_K(U_0)}\int_{z\in U_0}\omega(F(z))\,d\mu_K(z).$$

Then we have

$$Z_K(\omega)=\prod_{k=1}^{n-e_r}\frac{(n-k+1)}{1-q^{-\{\xi_k(s)+n-k+1\}}}\cdot E_K(\omega). \tag{3.11}$$

In the same way, if we take $\omega(F)$ as $\tilde{\phi}_0$ in Lemma 3.5, then we have

$$\tilde{Z}_K(\omega)=\prod_{k=1}^{n-e_1}\frac{(n-k+1)}{1-q^{-\{\tilde{\xi}_k(s)+n-k+1\}}}\cdot\tilde{E}_K(\omega), \tag{3.12}$$

in which we put

$$\tilde{E}_K(\omega)=\frac{1}{\tilde{\mu}_K(\tilde{U}_0)}\int_{\tilde{z}\in\tilde{U}_0}\omega(F(\tilde{z}))\,d\tilde{\mu}_K(\tilde{z}).$$

As we noted, U_0 is a $GL_n(\mathcal{O}_K)$-homogeneous space and $\tilde{U}_0$ is also a $GL_n^*(\mathcal{O}_K)$-homogeneous space, hence $U_0=\tilde{U}_0$. By Lemma 3.3, we have $E_K(\omega)=\tilde{E}_K(\omega)$. Therefore, our theorem follows immediately from (3.11) and (3.12).

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